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Letter to the Editor

Discrete system order reduction using multipoint step response matching

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Abstract

The proposed method of linear time invariant discrete system order reduction is based on multipoint step response matching for both pole and zero evaluation of the low order model. Depending on the number of zeros and poles of the low order model, the number of points are selected on the time axis of the unit step response such that the unknown poles and zeros can be determined by solving a set of nonlinear equations using Newton's method.

Keywords: Response; Discrete system; Order reduction; Pole

The purpose of this letter is to suggest a method of order reduction of linear time invariant discrete system, which strictly belongs to the category of step response matching with no restriction on the location of poles and zeros in original high order system (OHOS). In methods reported in [1,5,7], the poles and zeros of the low order model (LOM) are calculated using some error minimization criterion whereas in the proposed method, the LOM is determined by fitting its step response to the step response of the OHOS in the least square sense.

The proposed method is meant for only single input single output (SISO) discrete systems modelled in frequency domain i.e., using a z-transfer function. The reasons for choosing a method deviating from already existing well tried techniques are:

(1) The method is free from choosing poles and zeros using separate criteria, making it thereby theoretically robust.

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Table 1

Examples	Order of OHOS	Order of LOM	Least square error (J)	
			Using proposed methods	Using other methods
1	7	2	42.6168	—
		3	3.4860×10^{-01}	6.4906 [3]
2	8	2	5.9000×10^{-03}	3.5310×10^{-01} [1]
		3	2.9000×10^{-03}	3.9000×10^{-02} [1]
3	4	2	3.8000×10^{-03}	1.6500×10^{-02} [2]
4	8	2	6.2570×10^{-04}	3.6000×10^{-03} [6]
5	8	2	2.0000×10^{-04}	2.0000×10^{-04} [4]

(2) Unlike earlier standard methods of model order reduction like Pade Approximation, Markov and Moment matching, Routh approximation, etc, the proposed method takes into consideration the unit step response of the OHOS 'a priori' thereby opening an avenue for better response matching.

The method is described below. The proposed method is finally applied on a number of examples for both comparisons with other methods and studying the accuracy of the technique developed. The results are summarized in Table 1.

1. Problem formulation

For a given high-order discrete time system with transfer function $G_n(z)$ (i.e. OHOS), a rational of degree n, a reduced order system $G_r(z)$ (i.e., LOM) of degree $r < n$ is obtained by interpolating the step responses. The problem can be formulated as:

For any system of the form

$$G_p(z) = \frac{N_p(z)}{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_p)} = \alpha_0 + \sum_{i=1}^{p} \frac{\alpha_i}{(z-\lambda_i)}, \quad |\lambda_i| < 1 \tag{1}$$

with $N_p(z) = \sum_{i=0}^{p} c_i z^i$, a polynomial of degree less then or equal to p. The impulse response will have the terms $\alpha_0, g_0, g_1, \ldots$. with $g_k = \sum_{i=1}^{p} \alpha_i \lambda_i^k$.

Now as the unit step has a z-transform $\sum_{k=0}^{\infty} z^{-k} = z/(z-1)$, the response of (1) to a unit step input is

$$\hat{G}_p(z) = \frac{zG_p(z)}{z-1} = \frac{z\alpha_0}{z-1} + z^{-1}\sum_{k=0}^{\infty} \hat{g}_k z^{-k}, \quad \hat{g}_k = \sum_{l=0}^{k} g_l = \sum_{l=0}^{k}\sum_{i=0}^{p} \alpha_i \lambda_i^l, \tag{2}$$

where α_0 is the steady state and $\hat{g}_k$ represents the transient part of the response. Note that α_0 can be recovered from the step response by $\alpha_0 = (z-1)\hat{G}_p(z)/z|_{z=1} = G_p(1)$. Each $\alpha_i, i > 0$ is related to the step response $\alpha_i = (z-\lambda_i)\hat{G}_p(z)/z|_{z=\lambda_i}$.

Now let, $G_n(z)$ have the form of $G_p(z)$ given above with $p=n$ and let $G_r(z)$ have the same form with $p=r$, and with all the related symbols given an accent. The aim is to find $G_r(z)$ from $G_n(z)$ by matching the step responses of LOM and OHOS at '$2r$' points by taking the steady state matching in consideration. It is presumed here that all the poles of the OHOS are distinct. The method is described in the following section in detail.

2. Description of the method

The algorithm of the proposed method is detailed stepwise below:

Step 1(a): *Exact matching of steady state parts of the unit step responses of OHOS and LOM*. The first step is to consider the steady state part equal in the unit step responses of $G_n(z)$and $G_r(z)$.

Equating the steady state ($k_{ss} = \infty$) corresponds to fitting $\alpha_0 = \alpha_0'$, where α_0 is the steady state part in $G_n(z)$ and α_0' is the steady state part in $G_r(z)$ for a unit step input.

(b): *Fitting the transient part at '2r' matching points*. Now after exact steady state matching, the remaining transient part of both the systems (OHOS and LOM) are fitted at '$2r$' matching points, the first of which is $k_0 = 0$, and the remaining ones being in the transient region leads to the condition:

$$\hat{g}'_{k_j} = \hat{g}_{k_j}, \quad j = 0, 1, \ldots, 2r-1, \tag{3}$$

where $\hat{g}'_{k_j}$ is the transient part in the unit step response of LOM, and $\hat{g}_{k_j}$ is the transient part in the unit step response of OHOS.

While choosing the matching points, these have to be suitably spread so that major portion of the transient part of the response curve can be covered leading a good quality of response matching in terms of least squares error (LSE) i.e., 'J' as shown below in (4).

$$J = \sum_{k=0}^{N} (\hat{g}_k - \hat{f}_k)^2, \tag{4}$$

where $\hat{g}_k$ is the transient part in the unit step response of LOM and $\hat{f}_k$ is the transient part in the unit step response of OHOS.

Then condition (3) gives a system of '$2r$' nonlinear equations in the '$2r$' unknowns ($\alpha_1', \ldots, \alpha_r'$, $\lambda_1', \ldots, \lambda_r'$) (where $\lambda_1', \ldots, \lambda_r'$ are the poles and $\alpha_1', \ldots, \alpha_r'$ are the corresponding residues in unit step response of LOM). This system of equations is solved by Newton's method. In the proposed method, the poles of the reduced system are taken to be strictly distinct. When they are repeated, the Newton's method can be accordingly modified.

Step 2: *Determination of unknown coefficients of numerator of LOM*. Now after knowing $\alpha_1', \ldots, \alpha_r', \lambda_1', \ldots, \lambda_r'$, from Step 1, finally the coefficients $c_0, \ldots, c_r$ are obtained as follows. First note that from $\alpha_0 = \alpha_0'$ i.e. $G_n(1) = G_r(1)$, we get

$$\frac{c_0' + c_1' + \cdots + c_r'}{(1-\lambda_1')\cdots(1-\lambda_r')} = G_n(1). \tag{5}$$

The remaining r equations are found as $\alpha_i' = \hat{G}_r(z)(z-\lambda_i')/z|_{z=\lambda_i'}$, hence

$$\frac{c_0' + c_1'(\lambda_i') + \cdots + c_r'(\lambda_i')^r}{(\lambda_i'-1)\prod_{j\neq i}(\lambda_i'-\lambda_j')} = \alpha_i', \quad i,j=1,\ldots,r. \tag{6}$$

Since the λ_i' and α_i' are known from the previous step, this is a system of $r+1$ linear equations in the $r+1$ unknowns $c_0',\ldots,c_r'$ which is readily solved.

Using the above steps the proposed method is used to solve a number of examples and one of the example is given in the next section.

3. Examples

Although a number of examples were solved using the method described above, details in terms of transfer functions of OHOS and LOM along with response plots are given only for one example, where as results of other four examples are shown in Table 1, which indicates the least squares error (LSE) i.e., 'J', given in (4), between the unit step responses of original high order and reduced low order systems for each example.

Example 1. The example used by Chen et al. [1] is considered here and the transfer function of the original eighth-order system, $G_8(z)$ is

$$G_8(z) = \frac{0.4209z^7 + 0.2793z^6 - 0.0526z^5 + 0.038z^4 - 0.1291z^3 - 0.0656z^2 + 0.011z - 0.0015}{z^8 - 0.4209z^7 - 0.2793z^6 + 0.0526z^5 - 0.038z^4 + 0.1291z^3 + 0.0656z^2 - 0.011z + 0.0015}.$$

Transfer function of the reduced second-order system using the proposed method is

$$G_2(z) = \frac{0.4623z - 0.3063}{z^2 - 1.5299z + 0.6857}, \quad J = 5.90 \times 10^{-03}.$$

Transfer function of the reduced third-order system using the proposed method is

$$G_3(z) = \frac{0.4211z^2 - 0.1894z - 0.0759}{z^3 - 1.5011z^2 + 0.6538z + 0.0028}, \quad J = 2.9 \times 10^{-03}.$$

Transfer function of the reduced second-order system obtained by Chen et al. [1] is

$$G_2(z) = \frac{0.3975z - 0.318}{z^2 - 1.6025z + 0.682}, \quad J = 3.531 \times 10^{-01}.$$

Transfer function of the reduced third-order system obtained by Chen et al. [1] is

$$G_3(z) = \frac{15.948z^2 - 20.895z + 6.948}{35.944z^3 - 77.674z^2 + 58.516z - 14.786}, \quad J = 3.9 \times 10^{-02}.$$

The step responses between the original and reduced second-order systems are shown in Fig. 1 and the original and reduced third-order systems are shown in Fig. 2.

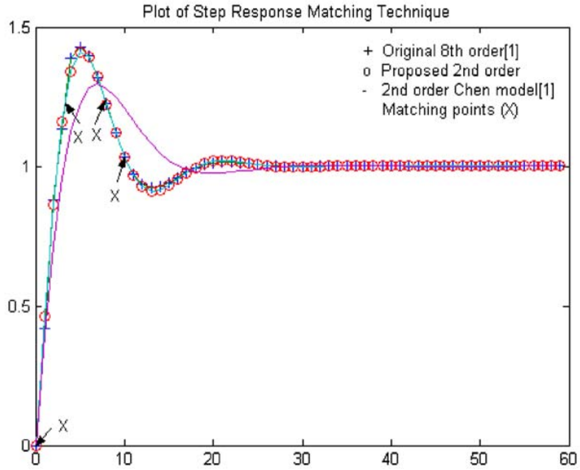

Fig. 1. Step response comparison of original and reduced second-order model of Example 1. Matching points: 0 3.3 7.8 9.7 Step response value: 0 1.2257 1.2436 1.0565.

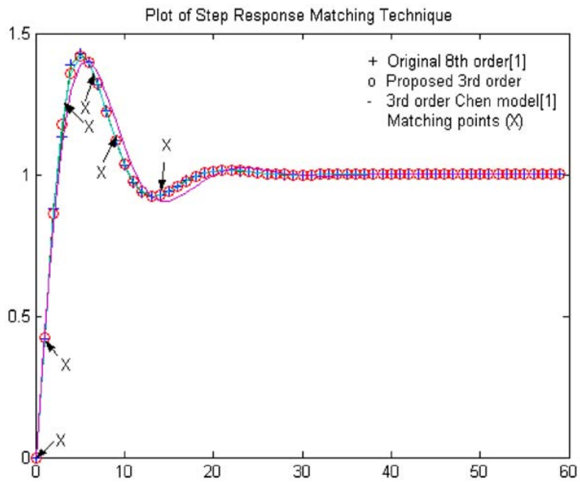

Fig. 2. Step response comparison of original and reduced third-order model of Example 1. Matching points : 0 1.0 3.5 6.1 8.8 14.2 Step response value: 0 0.4211 1.2829 1.3918 1.1404 0.9291.

4. Conclusions

After having used the proposed method to reduce OHOS of orders 8, 7 and 4 to LOM of orders 2 and 3, it could be observed as in Table 1, that the values of 'J' are better when compared to results of other workers [1,2,4,6] applied to the same systems. Besides improvement in the value of 'J', the method is free from selecting poles and zeros using different well known and tried techniques demanding more computational efforts. Inherent drawbacks of dominant pole retention are also absent in this method.

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